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Observations on Non-homogeneous Binary Quadratic Diophantine Equation

$$3x^2 + 10xy + 4y^2 - 4x + 2y - 7 = 0$$

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Abstract

This paper concerns with the problem of obtaining non-zero distinct integer solutions to non-homogeneous binary quadratic diophantine equation given by $3x^2 + 10xy + 4y^2 - 4x + 2y - 7 = 0$. A few interesting relations between the solutions are presented.

Keywords: Binary quadratic ,Non-homogeneous quadratic, Hyperbola, Integer solutions

Introduction

It is quite obvious that Diophantine equations, one of the areas of number theory, are rich in variety . In particular, the quadratic Diophantine equations in connection with geometrical figures occupy a pivotal role in the orbit of mathematics and have a wealth of historical significance. In this context, one may refer [1-14] for second degree Diophantine equations with two and three unknowns representing different geometrical figures.

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This paper aims at the problem of obtaining non-zero distinct integer solutions to non-homogeneous binary quadratic diophantine equation given by $3x^2+10xy+4y^2-4x+2y-7=0$. A few interesting relations between the solutions are presented.

Methodology

The non-homogeneous binary quadratic equation representing geometrically the hyperbola to be analyzed for its integer solutions is

$$3x^2+10xy+4y^2-4x+2y-7=0 \quad (1)$$

Introduction of the linear transformations

$$x=X-1,\ y=Y+1 \quad (2)$$

in (1) leads to the binary quadratic equation

$$3X^2+10XY+4Y^2-4=0 \quad (3)$$

Procedure 1

Treating (3) as a quadratic in X and solving for the same , we have

$$X=\frac{-5Y\pm\sqrt{13Y^2+12}}{3} \quad (4)$$

Let

$$\alpha^2=13Y^2+12 \quad (5)$$

The above equation is known as positive Pell equation whose least positive integer solution is given by

$$Y_0=1,\alpha_0=5$$

To obtain the other integer solutions to (5) , consider the Pell equation

$$\alpha^2=13Y^2+1$$

whose general solution $(\tilde{\alpha}_n,\tilde{Y}_n)$ is given by

$$\tilde{\alpha}_n=\frac{f_n}{2},\tilde{Y}_n=\frac{g_n}{2\sqrt{13}}$$

where

$$f_n=(649+180\sqrt{13})^{n+1}+(649-180\sqrt{13})^{n+1},$$
$$g_n=(649+180\sqrt{13})^{n+1}-(649-180\sqrt{13})^{n+1}.,n=-1,0,1,\ldots$$

Applying Brahmagupta lemma between (α_0,Y_0) & $(\tilde{\alpha}_n,\tilde{Y}_n)$,we get

$$\alpha_{n+1} = \frac{5}{2} f_n + \frac{\sqrt{13}}{2} g_n \ ,$$

$$Y_{n+1} = \frac{1}{2} f_n + \frac{5\sqrt{13}}{26} g_n \ .$$

From (4) ,we get

$$X_{n+1} = \frac{-5Y_{n+1} \pm \alpha_{n+1}}{3} = -\frac{2}{\sqrt{13}} g_n \ , -\frac{5}{3} f_n - \frac{19}{3\sqrt{13}} g_n$$

In view of (2) , we have the following two sets of integer solutions to (1):

Set 1

$$x_{n+1} = X_{n+1} - 1 = -\frac{2}{\sqrt{13}} g_n - 1,$$

$$y_{n+1} = Y_{n+1} + 1 = \frac{1}{2} f_n + \frac{5}{2\sqrt{13}} g_n + 1 \ , n = -1, 0, 1, \ldots$$

Set 2

$$x_{n+1} = X_{n+1} - 1 = -\frac{5}{3} f_n - \frac{19}{3\sqrt{13}} g_n - 1,$$

$$y_{n+1} = Y_{n+1} + 1 = \frac{1}{2} f_n + \frac{5}{2\sqrt{13}} g_n + 1, n = -1, 0, 1, \ldots$$

For the sake of clear understanding and brevity ,the relations between the solutions to (1) from Set 1 are exhibited below:

A few numerical solutions to (1) obtained from Set 1 are given as below:

$$x_0 = -1, y_0 = 2$$

$$x_1 = -721, y_1 = 1550$$

$$x_2 = -934561, y_2 = 2010602$$

The recurrence relations satisfied by the solutions x_{n+1}, y_{n+1} in Set 1 are respectively given by

$$x_{n+3} - 1298 x_{n+2} + x_{n+1} = 1296,$$

$$y_{n+3} - 1298 y_{n+2} + y_{n+1} = -1296, n = -1, 0, 1, \ldots$$

Some fascinating connections between the solutions in Set 1 are exhibited below:

(i) $4 y_{2n+2} + 5 x_{2n+2} + 5$ is a square multiple of 2

(ii) $649 x_{2n+2} - x_{2n+3} + 1368$ is a square multiple of 10

(iii) $x_{2n+1} - x_{2n+3} + 1440$ is a square multiple of 5

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(iv) $x_{2n+1} - 649x_{2n+2} + 72$ is asquare multiple of 10

(v) $y_{2n+3} - 1117y_{2n+2} + 1548$ is asquare multiple of 6

(vi) $4y_{2n+3} + 5585x_{2n+2} + 11777$ is a square multiple of product of two prime numbers 2 &1549

(vii) $(4y_{2n+2} + 5x_{2n+2} + 5) - 13(x_{n+1} + 1)^2 = 16$

(viii) $(4y_{n+1} + 5x_{n+1} + 1)^2 - 13(x_{n+1} + 1)^2 = 16$

(ix) $(y_{n+2} - 1117y_{n+1} + 1116)^2 - 13*(108)^2(x_{n+1} + 1)^2 = 4*(216)^2$

Procedure 2

Treating (3) as a quadratic in Y and solving for the same ,we have

$$Y = \frac{-5X \pm \sqrt{13X^2 + 16}}{4} \tag{6}$$

The choice

$$X = 4P \tag{7}$$

in (6) gives

$$Y = -5P \pm \sqrt{13P^2 + 1}$$

which is satisfied by

$$P_n = \frac{1}{2\sqrt{13}} g_n ,$$

$$Y_n = -\frac{5}{2\sqrt{13}} g_n \pm \frac{1}{2} f_n$$

In view of (7) & (2) , we get the following two sets of integer solutions to (1):

Set 3

$$x_n = 4P_n - 1 = \frac{2}{\sqrt{13}} g_n - 1,$$

$$y_n = 1 - \frac{5}{2\sqrt{13}} g_n + \frac{1}{2} f_n , n = 0,1,2,...$$

A few numerical solutions to (1) are given below:

$$x_0 = 719 , y_0 = -250$$

$$x_1 = 934559, y_1 = -325798$$

Set 4

$$x_n = \frac{2}{\sqrt{13}} g_n - 1,$$

$$y_n = 1 - \frac{5}{2\sqrt{13}} g_n - \frac{1}{2} f_n , n = 0,1,2,...$$

A few numerical solutions to (1) are as below:

$$x_0 = 719 , y_0 = -1548$$

$$x_1 = 934559 , y_1 = -2010600$$

Conclusion

In this paper ,an attempt has been made to obtain many non-zero distinct integer solutions to non-homogeneous binary quadratic Diophantine equation representing hyperbola given in title.The researchers may search for other choices of integer solutions to the considered hyperbola in this paper.

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